Dynamics of vortex and magnetic lines in ideal hydrodynamics and MHD

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Abstract

Vortex line and magnetic line representations are introduced for description of flows in ideal hydrodynamics and MHD, respectively. For incompressible fluids it is shown that the equations of motion for vorticity $\mathbf{\Omega}$ and magnetic field with the help of this transformation follow from the variational principle. By means of this representation it is possible to integrate the system of hydrodynamic type with the Hamiltonian $\mathcal{H} = \int |\mathbf{\Omega}| d\mathbf{r}$. It is also demonstrated that these representations allow to remove from the noncanonical Poisson brackets, defined on the space of divergence-free vector fields, degeneracy connected with the vorticity frozenness for the Euler equation and with magnetic field frozenness for ideal MHD. For MHD a new Weber type transformation is found. It is shown how this transformation can be obtained from the two-fluid model when electrons and ions can be considered as two independent fluids. The Weber type transformation for ideal MHD gives the whole Lagrangian vector invariant. When this invariant is absent this transformation coincides with the Clebsch representation analog introduced by Zakharov and Kuznetsov.

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1 Introduction

There are a large number of works devoted to the Hamiltonian description of the ideal hydrodynamics (see, for instance, the review [2] and the references therein). This question was first studied by Clebsch (a citation can be found in Ref. [3]), who introduced for nonpotential flows of incompressible fluids a pair of variables λ and μ (which later were called as the Clebsch variables). A fluid dynamics in these variables is such that vortex lines represent themselves intersection of surfaces $\lambda = \text{const}$ and $\mu = \text{const}$ and these quantities, being canonical conjugated variables, remain constant by fluid advection. However, these variables, as known (see, i.e.,[4]) describe only partial type of flows. If λ and μ are single-valued functions of coordinates then the linking degree of vortex lines characterizing by the Hopf invariant [5] occurs to be equal to zero. For arbitrary flows the Hamiltonian formulation of the equation for incompressible ideal hydrodynamics was given by V.I.Arnold [6, 7]. The Euler equations for the velocity curl $\mathbf{\Omega} = \operatorname{curl} \mathbf{v}$

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \operatorname{curl}\,[\mathbf{v} \times \mathbf{\Omega}], \quad \operatorname{div} \mathbf{v} = 0 \tag{1.1}$$

are written in the Hamiltonian form,

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \{\mathbf{\Omega}, \mathcal{H}\}, \tag{1.2}$$

by means of the noncanonical Poisson brackets [4]

$$\{F, G\} = \int \left(\mathbf{\Omega} \left[\operatorname{curl} \frac{\delta F}{\delta \mathbf{\Omega}} \times \operatorname{curl} \frac{\delta G}{\delta \mathbf{\Omega}} \right] \right) d\mathbf{r} \tag{1.3}$$

where the Hamiltonian

$$\mathcal{H}_h = -\frac{1}{2} \int \mathbf{\Omega} \Delta^{-1} \mathbf{\Omega} d^3\mathbf{r}, \tag{1.4}$$

coincides with the total fluid energy.

In spite of the fact that the bracket (1.3) allows to describe flows with arbitrary topology its main lack is a degeneracy. By this reason it is impossible to formulate the variational principle on the whole space $\mathcal{S}$ of divergence-free vector fields.

The cause of the degeneracy, namely, presence of Casimirs annulling the Poisson bracket, is connected with existence of the special symmetry formed the whole group - the relabeling group of Lagrangian markers (for details see the reviews [8, 2]). All known theorems about the vorticity conservation (the Ertel's, Cauchy's and Kelvin's theorems, the frozenness of vorticity and conservation of the topological Hopf invariant) are a sequence of this symmetry. The main of them is the frozenness of vortex lines into fluid. This is related to the local Lagrangian invariant – the Cauchy invariant. The physical meaning of this invariant consists in that any fluid particle remains all the time on its own vortex line.

The similar situation takes place also for ideal magneto-hydrodynamics (MHD) for barotropic fluids:

$$\rho_t + \nabla(\rho \mathbf{v}) = 0, \tag{1.5}$$

$$\mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v} = -\nabla w(\rho) + \frac{1}{4\pi\rho}[\operatorname{curl} \mathbf{h} \times \mathbf{h}], \tag{1.6}$$

$$\mathbf{h}_t = \mathrm{curl}[\mathbf{v} \times \mathbf{h}]. \tag{1.7}$$

Here ρ is a plasma density, $w(\rho)$ plasma entalpy, $\mathbf{v}$ and $\mathbf{h}$ are velocity and magnetic fields, respectively. As well known (see, for instance, [9]-[13]), the MHD equations possesses one important feature – frozenness of magnetic field into plasma which is destroyed only due to dissipation (by finite conductivity). For ideal MHD combination of the continuity equation (1.5) and the induction equation (1.7) gives the analog of the Cauchy invariant for MHD.

The MHD equations of motion (1.5-1.7) can be also represented in the Hamiltonian form,

$$\rho_t = \{\rho, \mathcal{H}\} \qquad \mathbf{h}_t = \{\mathbf{h}, \mathcal{H}\}, \qquad \mathbf{v}_t = \{\mathbf{v}, \mathcal{H}\}, \tag{1.8}$$

by means of the noncanonical Poisson brackets [14]:

$$\{F, G\} = \int \left(\frac{\mathbf{h}}{\rho} \cdot \left(\left[\mathrm{curl} \frac{\delta F}{\delta \mathbf{h}} \times \frac{\delta G}{\delta \mathbf{v}} \right] - \left[\mathrm{curl} \frac{\delta G}{\delta \mathbf{h}} \times \frac{\delta F}{\delta \mathbf{v}} \right] \right) \right) d^3\mathbf{r} + \tag{1.9}$$

$$+ \int \left(\frac{\mathrm{curl}\ \mathbf{v}}{\rho} \cdot \left[\frac{\delta F}{\delta \mathbf{v}} \times \frac{\delta G}{\delta \mathbf{v}} \right] \right) d^3\mathbf{r} + \int \left(\frac{\delta G}{\delta \rho} \nabla \left(\frac{\delta F}{\delta \mathbf{v}} \right) - \frac{\delta F}{\delta \rho} \nabla \left(\frac{\delta G}{\delta \mathbf{v}} \right) \right) d^3\mathbf{r}.$$

This bracket is also degenerated. For instance, the integral $\int (\mathbf{v}, \mathbf{h}) d\mathbf{r}$, which characterizes mutual linkage knottiness of vortex and magnetic lines, is one of the Casimirs for this bracket.

The analog of the Clebsch representation in MHD serves a change of variables suggested in 1970 by Zakharov and Kuznetsov [1]:

$$\mathbf{v} = \nabla \phi + \frac{[\mathbf{h} \times \mathrm{curl}\ \mathbf{S}]}{\rho}. \tag{1.10}$$

New variables (ϕ, ρ) and $\mathbf{h}, \mathbf{S}$ represent two pairs canonically conjugated quantities with the Hamiltonian coinciding with the total energy

$$\mathcal{H} = \int \left(\rho \frac{\mathbf{v}^2}{2} + \rho \varepsilon(\rho) + \frac{\mathbf{h}^2}{8\pi} \right) d\mathbf{r}.$$

In the present paper we suggest a new approach of the degeneracy resolution of the noncanonical Poisson brackets by introducing new variables, i.e., Lagrangian markers labeling each vortex lines for ideal hydrodynamics or magnetic lines in the MHD case.

The basis of this approach is the integral representation for the corresponding frozen-in field, namely, the velocity curl for the Euler equation and magnetic field for MHD. We introduce new objects, i.e., the vortex lines or magnetic lines and obtain the equations of motion for them. This description is a mixed Lagrangian-Eulerian description, when each vortex (or magnetic) line is enumerated by Lagrangian marker, but motion along the line is described in terms of the Eulerian variables. Such representation removes all degeneracy from the Poisson brackets connected with the frozenness, remaining the equations of motion to be gauge invariant with respect to re-parametrization of each line. Important, that the equations for line motion, as the equations for curve deformation, are transverse to the line tangent.

It is interesting that the line representation also solves another problem - the equations of line motion follow from the variational principle, being Hamiltonian.

This approach allows also simply enough to consider the limit of narrow vortex (or magnetic) lines. For two-dimensional flows in hydrodynamics this "new" description corresponds to the well-known fact, namely, to the canonical conjugation of x and y coordinates of vortices (see, for instance, [3]).

The Hamiltonian structure introduced makes it possible to integrate the three-dimensional Euler equation (1.2) with Hamiltonian $\mathcal{H} = \int |\mathbf{\Omega}| d\mathbf{r}$. In terms of the vortex lines the given Hamiltonian is decomposed into a set of Hamiltonians of noninteracting vortex lines. The dynamics of each vortex lines is, in turn, described by the equation of a vortex induction which can be reduced by the Hasimoto transformation [15] to the integrable one-dimensional nonlinear Schrodinger equation.

For ideal MHD a new representation - analog of the Weber transformation - is found. This representation contains the whole vector Lagrangian invariant. In the case of ideal hydrodynamics this invariant provides conservation of the Cauchy invariant and, as a sequence, all known conservation laws for vorticity (for details see the review [2]). It is important that all these conservation laws can be expressed in terms of observable variables. Unlike the Euler equation, these vector Lagrangian invariants for the MHD case can not be expressed in terms of density, velocity and magnetic field. It is necessary to tell that the analog of the Weber transformation for MHD includes the change of variables (1.10) as a partial case. The presence of these Lagrangian invariants in the transform provides topologically nontrivial MHD flows.

The Weber transform and its analog for MHD play a key role in constructing the vortex line (or magnetic line) representation. This representation is based on the property of frozenness. Just therefore by means of such transform the noncanonical Poisson brackets become non-degenerated in these variables and, as a result, the variational principle may be formulated. Another peculiarity of this representation is its locality, establishing the correspondence between vortex (or magnetic) line and vorticity (or magnetic field). This is a specific mapping, mixed Lagrangian-Eulerian, for which Jacobian of the mapping can not be equal to unity for incompressible fluids as it is for pure Lagrangian description.

2 General remarks

We start our consideration from some well known facts, namely, from the Lagrangian description of the ideal hydrodynamics.

In the Eulerian description for barotropic fluids, pressure $p = p(\rho)$, we have coupled equations - discontinuity equation for density ρ and the Euler equation for velocity:

$$\rho_t + \operatorname{div} \rho\mathbf{v} = 0, \tag{2.1}$$

$$\mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v} = -\nabla w(\rho), \qquad dw(\rho) = dp/\rho. \tag{2.2}$$

In the Lagrangian description each fluid particle has its own label. This is three-dimensional vector $\mathbf{a}$, so that particle position at time t is given by the function

$$\mathbf{x} = \mathbf{x}(\mathbf{a},t). \tag{2.3}$$

Usually initial position of particle serves the Lagrangian marker: $\mathbf{a} = \mathbf{x}(\mathbf{a},0)$.

In the Lagrangian description the Euler equation (2.2) is nothing more than the Newton equation:

$$\ddot{\mathbf{x}} = -\nabla w.$$

In this equation the second derivative with respect to time t is taken for fixed $\mathbf{a}$, but the r.h.s. of the equation is a function of t and $\mathbf{x}$. Excluding from the latter the x-dependence, the Euler equation takes the form:

$$\ddot{x}_i \frac{\partial x_i}{\partial a_k} = -\frac{\partial w(\rho)}{\partial a_k}, \tag{2.4}$$

where now all quantities are functions of t and $\mathbf{a}$.

In the Lagrangian description the continuity equation (2.1) is easily integrated and the density is given through the Jacobian of the mapping (2.3) $J = \det(\partial x_i/\partial a_k)$:

$$\rho = \frac{\rho_0(\mathbf{a})}{J}. \tag{2.5}$$

Now let us introduce a new vector,

$$u_k = \frac{\partial x_i}{\partial a_k} v_i\,, \tag{2.6}$$

which has a meaning of velocity in a new curvilinear system of coordinates or it is possible to say that this formula defines the transformation law for velocity components. It is worth noting that (2.6) gives the transform for the velocity $\mathbf{v}$ as a *co-vector*.

The straightforward calculation gives that the vector $\mathbf{u}$ satisfies the equation

$$\frac{du_k}{dt} = \frac{\partial}{\partial a_k}\left(\frac{\mathbf{v}^2}{2} - w\right). \tag{2.7}$$

In this equation the right-hand-side represents gradient relative to $\mathbf{a}$ and therefore the "transverse" part of the vector $\mathbf{u}$ will conserve in time. And this gives the Cauchy invariant:

$$\frac{d}{dt}\mathrm{curl}_{\;a}\,\mathbf{u} = 0, \tag{2.8}$$

or

$$\mathrm{curl}_a\,\mathbf{u} = \mathbf{I}. \tag{2.9}$$

If Lagrangian markers $\mathbf{a}$ are initial positions of fluid particles then the Cauchy invariant coincides with the initial vorticity: $\mathbf{I} = \mathbf{\Omega_0}(\mathbf{a})$. This invariant is expressed through instantaneous value of $\mathbf{\Omega}(\mathbf{x}, t)$ by the relation

$$\mathbf{\Omega_0}(\mathbf{a}) = J(\mathbf{\Omega}(\mathbf{x}, t)\nabla)\mathbf{a}(\mathbf{x}, t) \tag{2.10}$$

where $\mathbf{a} = \mathbf{a}(\mathbf{x},t)$ is inverse mapping to (2.3). Following from (2.10) relation for $\mathbf{B} = \mathbf{\Omega}/\rho$,

$$B_{0i}(a) = \frac{\partial a_i}{\partial x_k} B_k(x, t),$$

shows that, unlike velocity, $\mathbf{B}$ transforms as a vector.

By integrating the equation (2.7) over time t we arrive at the so-called Weber transformation

$$\mathbf{u}(\mathbf{a},t) = \mathbf{u}_0(\mathbf{a}) + \nabla_a \Phi, \tag{2.11}$$

where the potential Φ obeys the Bernoulli equation:

$$\frac{d\Phi}{dt} = \frac{\mathbf{v}^2}{2} - w(\rho) \tag{2.12}$$

with the initial condition: $\Phi|_{t=0} = 0$. For such choice of Φ a new function $\mathbf{u_0}(\mathbf{a})$ is connected with the "transverse" part of $\mathbf{u}$ by the evident relation

$$\mathrm{curl}_a\, \mathbf{u}_0(\mathbf{a}) = \mathbf{I}.$$

The Cauchy invariant $\mathbf{I}$ characterizes the vorticity frozenness into fluid. It can be got by standard way considering two equations - the equation for the quantity $\mathbf{B} = \mathbf{\Omega}/\rho$,

$$\frac{d\mathbf{B}}{dt} = (\mathbf{B}\nabla)\mathbf{v}, \tag{2.13}$$

and the equation for the vector $\delta\mathbf{x} = \mathbf{x}(\mathbf{a}+\delta\mathbf{a}) - \mathbf{x}(\mathbf{a})$ between two adjacent fluid particles:

$$\frac{d\delta\mathbf{x}}{dt} = (\delta\mathbf{x}\nabla)\mathbf{v}, \tag{2.14}$$

The comparison of these two equations shows that if initially the vectors $\delta\mathbf{x}$ are parallel to the vector $\mathbf{B}$, then they will be parallel to each other all time. This is nothing more than the statement of the vorticity frozenness into fluid. Each fluid particle remains all the time at its own vortex line. The combination of Eqs. (2.13) and (2.14) leads to the Cauchy invariant. To establish this fact it is enough to write down the equation for the Jacoby matrix $J_{ij} = \partial x_i/\partial a_j$ which directly follows from (2.14):

$$\frac{d}{dt}\frac{\partial a_i}{\partial x_k} = -\frac{\partial a_i}{\partial x_j}\frac{\partial v_j}{\partial x_k},$$

that in combination with Eq. (2.13) gives conservation of the Cauchy invariant (2.9).

If now one comes back to the velocity field $\mathbf{v}$ then by use of Eqs. (2.6) and (2.11) one can get that

$$\mathbf{v} = u_{0k}\nabla a_k + \nabla\Phi \tag{2.15}$$

where gradient is taken with respect to $\mathbf{x}$. Here the equation for potential Φ has the standard form of the Bernoulli equation:

$$\Phi_t + (\mathbf{v}\nabla)\Phi - \frac{\mathbf{v^2}}{2} + w(\rho) = 0.$$

It is interesting to note that relations (2.9), as equations for determination of $\mathbf{x}(\mathbf{a},t)$, unlike Eqs (2.7), are of the first order with respect to time derivative. This fact also reflects in the expression for velocity (2.15) which can be considered as a result of the partial integration of the equations of motion (2.7). Of course, the velocity field given by (2.15) contain two unknown functions: one is the whole vector $\mathbf{a}(\mathbf{x},t)$ and another

is the potential Φ. For incompressible fluids the latter is determined from the condition $\operatorname{div} \mathbf{v} = 0$. In this case the Bernoulli equation serves for determination of the pressure.

Another important moment connected with the Cauchy invariant is that it follows from the variational principle (written in terms of Lagrangian variables) as a sequence of relabelling symmetry remaining invariant the action (for details, see the reviews [8, 2]). Passing from Lagrangian to Hamiltonian in this description we have no any problems with the Poisson bracket. It is given by standard way and does not contain any degeneracy that the noncanonical Poisson brackets (1.3) and (1.9) have. One of the main purposes of this paper is to construct such new description of the Euler equation (as well as the ideal MHD) which, from one side, would allow to retain the Eulerian description, as maximally as possible, but, from another side, would exclude from the very beginning all remains from the gauge invariance of the complete Euler description connected with the relabeling symmetry.

As for MHD, this system in one point has some common feature with the Euler equation: it also possesses the frozenness property. The equation for $\mathbf{h}/\rho$ coincides with (2.13) and therefore dynamics of magnetic lines is very familiar to that for vortex lines of the Euler equation. However, this analogy cannot be continued so far because the equation of motion for velocity differs from the Euler equation by the presence of pondermotive force. This difference remains also for incompressible case.

3 Vortex line representation

Consider the Hamiltonian dynamics of the divergence-free vector field $\mathbf{\Omega}(\mathbf{r}, t)$, given by the Poisson bracket (1.3) with some Hamiltonian $\mathcal{H}$ [1]:

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \operatorname{curl}\left[\operatorname{curl}\frac{\delta \mathcal{H}}{\delta \mathbf{\Omega}} \times \mathbf{\Omega}\right]. \tag{3.1}$$

As we have said, the bracket (1.3) is degenerate, as a result of which it is impossible to formulate the variational principle on the entire space $\mathcal{S}$ of solenoidal vector fields. It is known [2] that Casimirs f, annulling Poisson brackets, distinguish in $\mathcal{S}$ invariant manifolds $\mathcal{M}_f$ (symplectic leaves) on each of which it is possible to introduce the standard Hamiltonian mechanics and accordingly to write down a variational principle. We shall show that solution of this problem for the equations (3.1) is possible on the base of the property of frozenness of the field $\mathbf{\Omega}(\mathbf{r}, t)$, which allows to resolve all constrains, stipulated by the Casimirs, and gives the necessary formulation of the variational principle.

To each Hamiltonian $\mathcal{H}$ - functional of $\mathbf{\Omega}(\mathbf{r}, t)$ - we associate the generalized velocity

$$\mathbf{v}(\mathbf{r}) = \operatorname{curl}\frac{\delta \mathcal{H}}{\delta \mathbf{\Omega}}. \tag{3.2}$$

However one should note that the generalized $\mathbf{v}(\mathbf{r})$ is defined up to addition of the vector parallel to $\mathbf{\Omega}$:

$$\operatorname{curl}\frac{\delta \mathcal{H}}{\delta \mathbf{\Omega}} \to \operatorname{curl}\frac{\delta \mathcal{H}}{\delta \mathbf{\Omega}} + \alpha\mathbf{\Omega},$$

[1]The Hamiltonian (1.4) corresponds to ideal incompressible hydrodynamics.

that in no way does change the equation for $\mathbf{\Omega}$. Under the condition $(\mathbf{\Omega}\cdot\nabla\alpha)=0$ a new generalized velocity will have zero divergence and the frozenness equation (3.1) can be written already for the new $\mathbf{v}(\mathbf{r})$. A gauge changing of the generalized velocity corresponds to some addition of a Casimir to the Hamiltonian :

$$\mathcal{H}\to\mathcal{H}+f;\qquad \{f,..\}=0.$$

Hence becomes clear that the transformation

$$\mathbf{x}=\mathbf{x}(\mathbf{a},t)$$

of the initial positions of fluid particles $\mathbf{x}(\mathbf{a},0)=\mathbf{a}$ by the generalized velocity field $\mathbf{v}(\mathbf{r})$ through solution of the equation

$$\dot{\mathbf{x}}=\mathbf{v}(\mathbf{x},t) \tag{3.3}$$

is defined ambiguously due to the ambiguous definition of $\mathbf{v}(\mathbf{r})$ by means of (3.2). Therefore using full Lagrangian description to the systems (3.1) becomes ineffective.

Now we introduce the following general expression for $\mathbf{\Omega}(\mathbf{r})$, which is gauge invariant and fixes all topological properties of the system that are determined by the initial field $\mathbf{\Omega}_0(\mathbf{a})$[16]:

$$\mathbf{\Omega}(\mathbf{r},t)=\int\delta(\mathbf{r}-\mathbf{R}(\mathbf{a},t))(\mathbf{\Omega}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a},t)d^3\mathbf{a}. \tag{3.4}$$

Here now

$$\mathbf{r}=\mathbf{R}(\mathbf{a},t) \tag{3.5}$$

does not satisfy any more the equation (3.3) and, consequently, the mapping Jacobian $J=\det||\partial\mathbf{R}/\partial\mathbf{a}||$ is not assumed to equal 1, as it was for full Lagrangian description of incompressible fluids.

It is easily to check that from condition $(\nabla_{\mathbf{a}}\mathbf{\Omega}_0(\mathbf{a}))=0$ it follows that divergence of (3.4) is identically equal to zero.

The gauge transformation

$$\mathbf{R}(\mathbf{a})\to\mathbf{R}(\tilde{\mathbf{a}}_{\Omega_0}(\mathbf{a})) \tag{3.6}$$

leaves this integral unchanged if $\tilde{\mathbf{a}}_{\Omega_0}$ is arisen from $\mathbf{a}$ by means of arbitrary nonuniform translations along the field line of $\mathbf{\Omega}_0(\mathbf{a})$. Therefore the invariant manifold $\mathcal{M}_{\Omega_0}$ of the space $\mathcal{S}$, on which the variational principle holds, is obtained from the space $\mathcal{R}:\mathbf{a}\to\mathbf{R}$ of arbitrary continuous one-to-one three-dimensional mappings identifying $\mathcal{R}$ elements that are obtained from one another with the help of the gauge transformation (3.6) with a fixed solenoidal field $\mathbf{\Omega}_0(\mathbf{a})$.

The integral representation for $\mathbf{\Omega}$ (3.4) is another formulation of the frozenness condition - after integration of the relation (3.4) over area σ, transverse to the lines of $\mathbf{\Omega}$, follows that the flux of this vector remains constant in time:

$$\int_{\sigma(t)}(\mathbf{\Omega},d\mathbf{S}_{\mathbf{r}})=\int_{\sigma(0)}(\mathbf{\Omega_0},d\mathbf{S}_{\mathbf{a}}).$$

Here $\sigma(t)$ is the image of $\sigma(0)$ under the transformation (3.5).

It is important also that $\mathbf{\Omega_0}(\mathbf{a})$ can be expressed explicitly in terms of the instantaneous value of the vorticity and the mapping $\mathbf{a}=\mathbf{a}(\mathbf{r},t)$, inverse to (3.5). By integrating over the variables $\mathbf{a}$ in the relation (3.4),

$$\mathbf{\Omega}(\mathbf{R})=\frac{(\mathbf{\Omega}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a})}{\det||\partial\mathbf{R}/\partial\mathbf{a}||}, \tag{3.7}$$

where $\mathbf{\Omega_0}(\mathbf{a})$ can be represented in the form:

$$\mathbf{\Omega_0}(\mathbf{a}) = \det||\partial\mathbf{R}/\partial\mathbf{a}||(\mathbf{\Omega}(\mathbf{r})\nabla)\mathbf{a}. \tag{3.8}$$

This formula is nothing more than the Cauchy invariant (2.9). We note that according to Eq. (3.7) the vector $\mathbf{b} = (\mathbf{\Omega}_0(\mathbf{a})\nabla_\mathbf{a})\mathbf{R}(\mathbf{a})$ is tangent to $\mathbf{\Omega}(\mathbf{R})$. It is natural to introduce parameter s as an arc length of the initial vortex lines $\mathbf{\Omega_0}(\mathbf{a})$ so that

$$\mathbf{b} = \Omega_0(\nu)\frac{\partial\mathbf{R}}{\partial s}.$$

In this expression Ω_0 depends on the transverse parameter ν labeling each vortex line. In accordance with this, the representation (3.4) can be written in the form

$$\mathbf{\Omega}(\mathbf{r},t) = \int \Omega_0(\nu)d^2\nu \int \delta(\mathbf{r} - \mathbf{R}(s,\nu,t))\frac{\partial\mathbf{R}}{\partial s}ds, \tag{3.9}$$

whence the meaning of the new variables becomes clearer: To each vortex line with index ν there is associated the closed curve

$$\mathbf{r} = \mathbf{R}(s,\nu,t),$$

and the integral (3.9) itself is a sum over vortex lines. We notice that the parametrization by introduction of s and ν is local. Therefore as global the representation (3.9) can be used only for distributions with closed vortex lines.

To get the equation of motion for $\mathbf{R}(\nu,s,t)$ the representation (3.9) (in the general case - (3.4)) must be substituted in the Euler equation (3.1) and then a Fourier transform with respect to spatial coordinates performed. As a result of simple integration one can obtain:

$$\left[\mathbf{k} \times \int \Omega_0(\nu)d^2\nu \int ds e^{-i\mathbf{k}\mathbf{R}}[\mathbf{R}_s \times \{\mathbf{R}_t(\nu,s,t) - \mathbf{v}(\mathbf{R},t)\}]\right] = 0.$$

This equation can be resolved by putting integrand equal identically to zero:

$$[\mathbf{R}_s \times \mathbf{R}_t(\nu,s,t)] = [\mathbf{R}_s \times \mathbf{v}(\mathbf{R},t)]. \tag{3.10}$$

With this choice there remains the freedom in both changing the parameter s and relabelling the transverse coordinates ν. In the general case of arbitrary topology of the field $\mathbf{\Omega_0}(\mathbf{a})$ the vector $\mathbf{R}_s$ in the equation (3.10) must be replaced by the vector $\mathbf{b} = (\mathbf{\Omega}_0(\mathbf{a})\nabla_\mathbf{a})\mathbf{R}(\mathbf{a},t)$. Notice that, as it follows from (3.10) and (3.7), a motion of a point on the manifold $\mathcal{M}_{\Omega_0}$ is determined only by the transverse to $\mathbf{\Omega}(\mathbf{r})$ component of the generalized velocity.

The obtained equation (3.10) is the equation of motion for vortex lines. In accordance with (3.10) the evolution of each vector $\mathbf{R}$ is principally transverse to the vortex line. The longitudinal component of velocity does not effect on the line dynamics.

The description of vortex lines with the help of equations (3.9) and (3.10) is a mixed Lagrangian-Eulerian one: The parameter ν has a clear Lagrangian origin whereas the coordinate s remains Eulerian.

4 Variational principle

The key observation for formulation of the variational principle is that the following general equality holds for functionals that depend only on $\mathbf{\Omega}$:

$$\left[\mathbf{b} \times \mathrm{curl}\left(\frac{\delta F}{\delta \mathbf{\Omega}(\mathbf{R})}\right)\right] = \frac{\delta F}{\delta \mathbf{R}(\mathbf{a})}\Big|_{\mathbf{\Omega}_0}. \tag{4.1}$$

For this reason, the right-hand-side of (3.10) equals the variational derivative $\delta\mathcal{H}/\delta\mathbf{R}$:

$$[(\mathbf{\Omega}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a}) \times \mathbf{R}_t(\mathbf{a})] = \frac{\delta\mathcal{H}\{\mathbf{\Omega}\{\mathbf{R}\}\}}{\delta\mathbf{R}(\mathbf{a})}\Big|_{\mathbf{\Omega}_0}. \tag{4.2}$$

It is not difficult to check now that the equation (4.2) described dynamics of vortex line is equivalent to the requirement of extremum of the action ($\delta S = 0$) with the Lagrangian [16]

$$\mathcal{L} = \frac{1}{3}\int d^3\mathbf{a}([\mathbf{R}_t(\mathbf{a}) \times \mathbf{R}(\mathbf{a})] \cdot (\mathbf{\Omega}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a})) - \mathcal{H}(\{\mathbf{\Omega}\{\mathbf{R}\}\}). \tag{4.3}$$

Thus, we have introduced the variational principle for the Hamiltonian dynamics of the divergence-free vector field topologically equivalent to $\mathbf{\Omega}_0(\mathbf{a})$.

Let us discuss some properties of the equations of motion (4.2), which are associated with excess parametrization of elements of $\mathcal{M}_{\Omega_0}$ by objects from $\mathcal{R}$. We want to pay attention to the fact that From Eq. (4.1) follows the property that the vector $\mathbf{b}$ and $\delta F/\delta\mathbf{R}(\mathbf{a})$ are orthogonal for all functionals defined on $\mathcal{M}_{\Omega_0}$. In other words the variational derivative of the gauge-invariant functionals should be understood (specifically, in (4.1)) as

$$\hat{P}\frac{\delta F}{\delta\mathbf{R}(\mathbf{a})},$$

where $\hat{P}_{ij} = \delta_{ij} - \tau_i\tau_j$ is a projector and $\tau = \mathbf{b}/|\mathbf{b}|$ a unit tangent (to vortex line) vector. Using this property as well as the transformation formula (4.1) it is possible, by a direct calculation of the bracket (1.3), to obtain the Poisson bracket (between two gauge-invariant functionals) expressed in terms of vortex lines:

$$\{F, G\} = \int \frac{d^3\mathbf{a}}{|\mathbf{b}|^2}\left(\mathbf{b} \cdot \left[\hat{P}\frac{\delta F}{\delta\mathbf{R}(\mathbf{a})} \times \hat{P}\frac{\delta G}{\delta\mathbf{R}(\mathbf{a})}\right]\right). \tag{4.4}$$

The new bracket (4.4) does not contain variational derivatives with respect to $\mathbf{\Omega}_0(\mathbf{a})$. Therefore, with respect to the initial bracket the Cauchy invariant $\mathbf{\Omega}_0(\mathbf{a})$ is a Casimir fixing the invariant manifolds $\mathcal{M}_{\Omega_0}$ on which it is possible to introduce the variational principle (4.3).

In the case of the hydrodynamics of a superfluid liquid a Lagrangian of the form (4.3) was apparently first used by Rasetti and Regge [17] to derive an equation of motion, identical to Eq. (3.10), but for a separate vortex filament. Later, on the base of the results [17], Volovik and Dotsenko Jr. [18] obtained the Poisson bracket between the coordinates of the vortices and the velocity components for a continuous distribution of vortices. The expression for these brackets can be extracted without difficulty from the general form for the Poisson brackets (4.4) . However, the noncanonical Poisson brackets obtained in

[17, 18] must be used with care. Their direct application gives for the equation of motion of the coordinate of a vortex filament an answer that is not gauge-invariant. For a general variation, which depends on time, additional terms describing flow along a vortex appear in the equation of motion. For this reason, the dynamics of curves (including vortex lines) is in principle "transverse" with respect to the curve itself.

We note that for two-dimensional (in the $x-y$ plane) flows the variational principle for action with the Lagrangian (4.3) leads to the well-known fact that $X(\nu, t)$- and $Y(\nu, t)$-coordinates of each vortex are canonically conjugated quantities (see [3]).

5 Integrable hydrodynamics

Now we present an example of the equations of the hydrodynamic type (3.1), for which transition to the representation of vortex lines permits to establish of the fact of their integrability [16].

Consider the Hamiltonian

$$\mathcal{H}\{\mathbf{\Omega}(\mathbf{r})\} = \int |\mathbf{\Omega}| d\mathbf{r} \tag{5.1}$$

and the corresponding equation of frozenness (3.1) with the generalized velocity

$$\mathbf{v} = \operatorname{curl}(\mathbf{\Omega}/\Omega).$$

We assume that vortex lines are closed and apply the representation (3.9). Then due to (3.7) the Hamiltonian in terms of vortex lines is decomposed as a sum of Hamiltonians of vortex lines:

$$\mathcal{H}\{\mathbf{R}\} = \int |\Omega_0(\nu)| d^2\nu \int \left| \frac{\partial \mathbf{R}}{\partial s} \right| ds. \tag{5.2}$$

The standing here integral over s is the total length of a vortex line with index ν. According to (4.2), with respect to these variables the equation of motion for the vector $\mathbf{R}(\nu, s)$ is local, it does not contain terms describing interaction with other vortices:

$$\eta[\tau \times \mathbf{R}_t(\nu, s, t)] = [\tau \times [\tau \times \tau_s]]. \tag{5.3}$$

Here $\eta = \operatorname{sign}(\Omega_0)$, $\tau = \mathbf{R}_s/|\mathbf{R}_s|$ is the unit vector tangent to the vortex line.

This equation is invariant against changes $s \longrightarrow \tilde{s}(s, t)$. Therefore the equation (5.3) can be resolved relative to $\mathbf{R}_t$ up to a shift along the vortex line – the transformation unchanged the vorticity $\mathbf{\Omega}$. This means that to find $\mathbf{\Omega}$ it is enough to have one solution of the equation

$$\eta |\mathbf{R}_s| \mathbf{R}_t = [\tau \times \tau_s] + \beta \mathbf{R}_s, \tag{5.4}$$

which follows from (5.3) for some value of β. Arisen from here equation for τ as a function of filament length l ($dl = |\mathbf{R}_s| ds$) and time t (by choosing a new value $\beta = 0$) reduces to the integrable one-dimensional Landau-Lifshits equation for a Heisenberg ferromagnet:

$$\eta \frac{\partial \tau}{\partial t} = \left[\tau \times \frac{\partial^2 \tau}{\partial l^2} \right].$$

This equation is gauge-equivalent to the 1D nonlinear Schrödinger equation [19] and, for instance, can be reduced to the NLSE by means of the Hasimoto transformation [15]:

$$\psi(l,t) = \kappa(l,t) \cdot \exp(i \int^l \chi(\tilde{l},t) d\tilde{l}),$$

where $\kappa(l,t)$ is a curvature and $\chi(l,t)$ the line torsion.

The considered system with the Hamiltonian (5.1) has direct relation to hydrodynamics. As known (see the paper [15] and references therein), the local approximation for thin vortex filament (under assumption of smallness of the filament width to the characteristic longitudinal scale) leads to the Hamiltonian (5.2) but only for one separate line. Respectively, the equation (3.1) with the Hamiltonian (5.1) can be used for description of motion of a few number of vortex filaments, thickness of which is small compared with a distance between them. In this case (nonlinear) dynamics of each filament is independent upon neighbor behavior. In the framework of this model singularity appearance (intersection of vortices) is of an inertial character very similar to the wave breaking in gas-dynamics. Of course, this approximation does not work on distances between filaments comparable with filament thickness.

It should be noted also that for the given approximation the Hamiltonian of vortex line is proportional to the filament line whence its conservation follows that, however, in no cases is adequate to behavior of vortex filaments in turbulent flows where usually process of vortex filament stretching takes place. It is desirable to have the better model free from this lack. A new model must necessarily describe nonlocal effects.

In addition we would like to say that the list of equations (3.1) which can be integrated with the help of representation (3.9) is not exhausted by (5.1). So, the system with the Hamiltonian

$$\mathcal{H}_\chi\{\mathbf{\Omega}(\mathbf{r})\} = \int |\mathbf{\Omega}|\chi d\mathbf{r} \tag{5.5}$$

is gauge equivalent to the modified KdV equation

$$\psi_t + \psi_{lll} + \frac{3}{2}|\psi|^2\psi_l = 0 \ -$$

the second one after NLSE in the hierarchy generated by Zakharov-Shabat operator. As against previous model (5.1) some physical application of (5.5) has not yet been found.

6 Lagrangian description of MHD

Consider now how the relabelling symmetry works in the ideal MHD. First, rewrite equations of motion (1.5-1.7) in the Lagrangian representation by introducing markers $\mathbf{a}$ for fluid particles

$$\mathbf{x} = \mathbf{x}(\mathbf{a}, t)$$

with

$$\mathbf{v}(\mathbf{x}, t) = \dot{\mathbf{x}}(\mathbf{a}, t).$$

In this case the continuity equation (1.5) and the equation for magnetic field (1.7) can be integrated. The density and the magnetic field are expressed in terms of the Jacoby

matrix by means of Eq. (2.5) and by the equation

$$B_i(x,t) = \frac{\partial x_i}{\partial a_k} B_{0k}(a)\,, \tag{6.1}$$

where $\mathbf{B} = \mathbf{h}/\rho$. In the latter transformation the Jacoby matrix serves the evolution operator for vector $\mathbf{B}$. The vector $\mathbf{B}$, in turn, transforms as a vector.

In terms of Lagrangian variables the equation of motion (1.6) is written as follows

$$\frac{\partial x_i}{\partial a_k}\ddot{x}_i = -\frac{\partial w(\rho)}{\partial a_k} + \frac{J}{4\pi\rho_0(\mathbf{a})}[\mathrm{curl}\,\mathbf{h}\times\mathbf{h}]_i\frac{\partial x_i}{\partial a_k} \tag{6.2}$$

With the help of relation (6.1) and Eq. (2.7) the vector $\mathbf{u}$ given by (2.6) will satisfy the equation

$$\frac{d\mathbf{u}}{dt} = \nabla\left(\frac{\mathbf{v}^2}{2} - w\right) - \frac{1}{4\pi}\left[\mathbf{B}_0(\mathbf{a})\times\mathrm{curl}_a\mathbf{H}\right]. \tag{6.3}$$

Here vector $\mathbf{B}_0(\mathbf{a}) = \mathbf{h}_0(\mathbf{a})/\rho_0(\mathbf{a})$ is a Lagrangian invariant and $\mathbf{H}$ represents the co-adjoint transformation of the magnetic field, analogous to (2.6):

$$H_i(a,t) = \frac{\partial x_m}{\partial a_i} h_m(x,t).$$

Now by analogy with (2.7) and (2.11), integration of Eq.(6.3) over time leads to the Weber type transformation:

$$\mathbf{u}(\mathbf{a},t) = \mathbf{u}_0(\mathbf{a}) + \nabla_a\Phi + \left[\mathbf{B}_0(\mathbf{a})\times\mathrm{curl}_a\tilde{\mathbf{S}}\right]. \tag{6.4}$$

Here $\mathbf{u}_0(\mathbf{a})$ is a new Lagrangian invariant which can be chosen as pure transverse, namely, with $\mathrm{div}_a\,\mathbf{u}_0 = 0$. This new Lagrangian invariant cannot be expressed through the observed physical quantities such as magnetic field, velocity and density. In spite of this fact, as it will be shown in the next section, the vector Lagrangian invariant $\mathbf{u}_0(\mathbf{a})$ has a clear physical meaning. As far as new variables Φ and $\tilde{\mathbf{S}}$, they obey the equations:

$$\begin{aligned}\frac{d\Phi}{dt} &= \frac{\mathbf{v}^2}{2} - w,\\ \frac{d\tilde{\mathbf{S}}}{dt} &= -\frac{\mathbf{H}}{4\pi} + \nabla_a\psi.\end{aligned}$$

The transformation (6.4) for velocity $\mathbf{v}(\mathbf{x},t)$ takes the form:

$$\mathbf{v} = u_{0k}(\mathbf{a})\nabla a_k + \nabla\Phi + \left[\frac{\mathbf{h}}{\rho}\times\mathrm{curl}\,\mathbf{S}\right] \tag{6.5}$$

where $\mathbf{S}$ is the vector $\tilde{\mathbf{S}}$ transformed by means of the rule (2.6):

$$S_i(x,t) = \frac{\partial a_k}{\partial x_i}\tilde{S}_k(a,t).$$

In Eulerian description Φ satisfies the Bernoulli equation

$$\frac{\partial\Phi}{\partial t} + (\mathbf{v}\nabla)\Phi - \frac{\mathbf{v}^2}{2} + w = 0 \tag{6.6}$$

and equation of motion for $\mathbf{S}$ is of the form:

$$\frac{\partial \mathbf{S}}{\partial t} + \frac{\mathbf{h}}{4\pi} - [\mathbf{v} \times \mathrm{curl}\mathbf{S}] + \nabla \psi_1 = 0. \tag{6.7}$$

For $\mathbf{u}_0 = 0$ the transformation (6.5) was introduced for ideal MHD by Zakharov and Kuznetsov in 1970 [1]. In this case magnetic field $\mathbf{h}$ and vector $\mathbf{S}$ as well as Φ and ρ are two pairs of canonically conjugated variables. It is interesting to note that in the canonical case the equations of motion for $\mathbf{S}$ and Φ obtained in [1] coincide with (6.6) and (6.7). However, the canonical parametrization describes partial type of flows, in particular, it does not describe topological nontrivial flows for which mutual knottiness between magnetic and vortex lines is not equal to zero. This topological characteristics is given by the integral $\int(\mathbf{v}, \mathbf{h})d\mathbf{x}$. Only when $\mathbf{u}_0 \neq 0$ this integral takes non-zero values.

7 Frozen-in MHD fields

To clarify meaning of new Lagrangian invariant $\mathbf{u_0}(\mathbf{a})$ we remind that the MHD equations (1.5-1.7) can be obtained from two-fluid system where electrons and ions are considered as two separate fluids interacting each other by means of self-consistent electromagnetic field. The MHD equations follow from two-fluid equations in the low-frequency limit when characteristic frequencies are less than ion gyro-frequency. The latter assumes i) neglecting by electron inertia, ii) smallness of electric field with respect to magnetic field, and iii) charge quasi-neutrality. We write down at first some intermediate system called often as MHD with dispersion [20]:

$$\mathrm{curl}\ \mathrm{curl}\mathbf{A} = \frac{4\pi e}{c}(n_1\mathbf{v}_1 - n_2\mathbf{v}_2), \tag{7.1}$$

$$(\partial_t + \mathbf{v}_1\nabla)m\mathbf{v}_1 = \frac{e}{c}\left(-\mathbf{A}_t + [\mathbf{v}_1 \times \mathrm{curl}\ \mathbf{A}]\right) - \nabla\frac{\partial\varepsilon}{\partial n_1}, \tag{7.2}$$

$$0 = -\frac{e}{c}\left(-\mathbf{A}_t + [\mathbf{v}_2 \times \mathrm{curl}\ \mathbf{A}]\right) - \nabla\frac{\partial\varepsilon}{\partial n_2}. \tag{7.3}$$

In these equations $\mathbf{A}$ is the vector potential so that the magnetic field $\mathbf{h} = \mathrm{curl}\mathbf{A}$ and electric field $\mathbf{E} = -\frac{\mathbf{1}}{\mathbf{c}}\mathbf{A_t}$. This system is closed by two continuity equations for ion density n_1 and electron density n_2:

$$n_{1,t} + \nabla(n_1\mathbf{v}_1) = 0, \qquad n_{2,t} + \nabla(n_2\mathbf{v}_2) = 0. \tag{7.4}$$

In this system $\mathbf{v}_{1,2}$ are velocities of ion and electron fluids, respectively. The first equation of this system is a Maxwell equation for magnetic field in static limit. The second equation is equation of motion for ions. The next one is equation of motion for electrons in which we neglect by electron inertia. By means of the latter equation one can obtain the equation of frozenness of magnetic field into electron fluid (this is another Maxwell equation):

$$\mathbf{h}_t = \mathrm{curl}[\mathbf{v}_2 \times \mathbf{h}].$$

Applying the operator div to (7.1) gives with account of continuity equations the quasi-neutrality condition: $n_1 = n_2 = n$. Next, by excluding n_2 and $\mathbf{v}_2$ we have finally the MHD equations with dispersion in its standard form [20]:

$$(\partial_t + \mathbf{v}\nabla)m\mathbf{v} = -\nabla w(n) + \frac{1}{4\pi n}[\text{curl } \mathbf{h} \times \mathbf{h}], \qquad w(n) = \frac{\partial}{\partial n}\varepsilon(n,n),$$

$$n_t + \nabla(n\mathbf{v}) = 0, \qquad \mathbf{h}_t = \text{curl}\left[\left(\mathbf{v} - \frac{c}{4\pi en}\text{curl } \mathbf{h}\right) \times \mathbf{h}\right], \tag{7.5}$$

where $\mathbf{v}_1 = \mathbf{v}$, and $\varepsilon(n,n)$ is internal energy density so that $w(n)$ is entalpy per one pair ion-electron. The classical MHD follows from this system in the limit when the last term $c/(4\pi en)\text{curl}\,\mathbf{h}$ in equation (7.5) should be neglected with respect to $\mathbf{v}$. At the same time, the vector potential $\mathbf{A}$ must be larger characteristic values of $(mc/e)\mathbf{v}$ in order to provide inertia and magnetic terms in Eq. (7.2) being of the same order of magnitude. Both requirements are satisfied if $\epsilon = c/(\omega_{pi}L) << 1$ where L is a characteristic scale of magnetic field variation and $\omega_{pi} = \sqrt{4\pi ne^2/m}$ is ion plasma frequency.

Unlike MHD equations (1.5-1.7), the given system has two frozen-in fields. These are the field $\mathbf{\Omega}_2 = -\frac{e}{mc}\mathbf{h}$ frozen into electron fluid and the field

$$\mathbf{\Omega}_1 = \text{curl}(\mathbf{v} + \frac{e}{mc}\mathbf{A}) = \mathbf{\Omega} - \mathbf{\Omega}_2$$

frozen into ion component:

$$\mathbf{\Omega}_{1t} = \text{curl}\left[\mathbf{v} \times \mathbf{\Omega}_1\right],$$

$$\mathbf{\Omega}_{2t} = \text{curl}\left[\mathbf{v}_2 \times \mathbf{\Omega}_2\right]$$

where

$$\mathbf{v}_2 = \mathbf{v} - \frac{c}{4\pi en}\text{curl } \mathbf{h}.$$

Hence for both fields one can construct two Cauchy invariants by the same rule (2.9) as for ideal hydrodynamics:

$$\mathbf{\Omega_{10}}(\mathbf{a}) = J_1(\mathbf{\Omega_1}(\mathbf{x},t)\nabla)\mathbf{a}(\mathbf{x},t) \tag{7.6}$$

where $\mathbf{a}(\mathbf{x},t)$ is inverse mapping to $\mathbf{x} = \mathbf{x}_1(\mathbf{a},t)$ which is solution of the equation $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x},t)$;

$$\mathbf{\Omega_{20}}(\mathbf{a}_2) = J_2(\mathbf{\Omega_2}(\mathbf{x},t)\nabla)\mathbf{a}_2(\mathbf{x},t) \tag{7.7}$$

with $\mathbf{a}_2(\mathbf{x},t)$ inverse to the mapping $\mathbf{x} = \mathbf{x}_2(\mathbf{a}_2,t)$ and $\dot{\mathbf{x}} = \mathbf{v_2}(\mathbf{x},t)$.

In order to get the corresponding Weber transformation for MHD as a limit of the system it is necessary to introduce two momenta for ion and electron fluids:

$$\mathbf{p}_1 = m\mathbf{v} + \frac{e}{c}\mathbf{A} \tag{7.8}$$

$$\mathbf{p}_2 = -\frac{e}{c}\mathbf{A}. \tag{7.9}$$

In these expressions the terms containing the vector potential are greater sum of $\mathbf{p}_1$ and $\mathbf{p}_2$ in parameter ϵ. For each momentum in Lagrangian representation one can get equations, analogous to (2.4), (2.7):

$$\frac{\partial x_k}{\partial a_{1i}}\frac{dp_{1k}}{dt} = -p_{1k}\frac{\partial v_k}{\partial a_{1i}} + \frac{\partial}{\partial a_{1i}}\left(-\frac{\partial\varepsilon}{\partial n_1} + \frac{e}{c}(\mathbf{v}\cdot\mathbf{A}) + m\frac{v^2}{2}\right) \tag{7.10}$$

$$\frac{\partial x_k}{\partial a_{2i}}\frac{dp_{2k}}{dt} = -p_{2k}\frac{\partial v_{2k}}{\partial a_{2i}} + \frac{\partial}{\partial a_{2i}}\left(-\frac{\partial\varepsilon}{\partial n_2} - \frac{e}{c}(\mathbf{v}_2\cdot\mathbf{A})\right). \tag{7.11}$$

By introducing the vector $\tilde{\mathbf{p}}$ for each type of fluids, by the same rule as (2.6),

$$\tilde{p}_i = \frac{\partial x_k}{\partial a_i} p_k,$$

after integration over time of equations of motion for $\tilde{\mathbf{p}}$ one can arrive at two Weber transformations for each momentum:

$$\mathbf{p}_1 = \tilde{p}_{1i}(a_1)\nabla a_{1i} + \nabla\Phi_1, \tag{7.12}$$

$$\mathbf{p}_2 = \tilde{p}_{2i}(a_2)\nabla a_{2i} + \nabla\Phi_2. \tag{7.13}$$

In the limit $\epsilon \to 0$ the markers $\mathbf{a_1}$ and $\mathbf{a}_2$ can be put approximately equal. This means that their difference will be small:

$$\mathbf{a}_2 - \mathbf{a_1} = \mathbf{d} \ \sim \epsilon.$$

Besides, due to charge quasi-neutrality, Jacobians with respect to a_1 and a_2 must be equal each other (here we put $n_{10}(\mathbf{a_1}) = n_{20}(\mathbf{a}_2) = 1$ without loss of generality):

$$\det||\partial\mathbf{x}/\partial\mathbf{a_1}|| = \det||\partial\mathbf{x}/\partial\mathbf{a_2}||.$$

As a result, the infinitesimal vector $\mathbf{d}(\mathbf{a},t)$ relative to the argument $\mathbf{a}$ occurs divergence free: $\partial d_i/\partial a_i = 0$.

Then, summing (7.12) and (7.13) and considering the limit $\epsilon \to 0$, we obtain the Weber-type transformation coinciding with (6.4):

$$\mathbf{u}(\mathbf{a},t) = \mathbf{u}_0(\mathbf{a}) + \nabla_a\Phi + \left[\mathbf{B}_0(\mathbf{a}) \times \mathrm{curl}_a\,\tilde{\mathbf{S}}\right], \tag{7.14}$$

where vectors $\mathbf{u}_0(\mathbf{a})$ and $\tilde{\mathbf{S}}$ are expressed through the Lagrangian invariants $\tilde{\mathbf{p}}_1(\mathbf{a})$ and $\tilde{\mathbf{p}}_2(\mathbf{a})$ and displacement $\mathbf{d}$ between electron and ion by means of relations [21]:

$$\mathbf{u}(\mathbf{a},t) = \frac{1}{m}(\tilde{\mathbf{p}}_1(\mathbf{a}) + \tilde{\mathbf{p}}_2(\mathbf{a})),$$

$$\mathbf{d} = -\frac{mc}{e}\mathrm{curl}_a\tilde{\mathbf{S}}.$$

Important that in (7.14) all terms are of the same order of magnitude (zero order relative to ϵ). Curl of vectors $\tilde{\mathbf{p}}_1(\mathbf{a})$ and $\tilde{\mathbf{p}}_2(\mathbf{a}_2)$ yield the corresponding Cauchy invariants (7.6) and (7.7).

8 Relabeling symmetry in MHD

Now let us show how existence of new Lagrangian invariants corresponds to the relabeling symmetry.

Consider the MHD Lagrangian [2],

$$\mathcal{L}_* = \int \left(\rho\frac{\mathbf{v}^2}{2} - \rho\tilde{\varepsilon}(\rho) - \frac{\mathbf{h}^2}{8\pi}\right) d\mathbf{r},$$

where we neglect by contribution from electric field in comparison with that from magnetic field. Here $\tilde{\varepsilon}(\rho)$ is specific internal energy.

In terms of mapping $\mathbf{x}(\mathbf{a}, t)$ the Lagrangian $\mathcal{L}_*$ is rewritten as follows [22]:

$$\mathcal{L}_* = \int \frac{\dot{\mathbf{x}}^2}{2} d^3\mathbf{a} - \int \tilde{\varepsilon}(J_{\mathbf{x}}^{-1}(\mathbf{a}))d^3\mathbf{a} - \frac{1}{8\pi}\int \left(\frac{(\mathbf{h_0}(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{x}}{J_{\mathbf{x}}(\mathbf{a})}\right)^2 J_{\mathbf{x}}(\mathbf{a})d^3\mathbf{a}. \tag{8.1}$$

Here density and magnetic field are expressed by means of relations

$$\rho = 1/J_{\mathbf{x}}, \qquad \mathbf{h} = (\mathbf{h}_0(\mathbf{a})\nabla_a)\mathbf{x}/J_{\mathbf{x}},$$

and

$$J_x(\mathbf{a}, t) = \det||\partial\mathbf{x}/\partial\mathbf{a}||$$

is the Jacobian of mapping $\mathbf{x} = \mathbf{x}(\mathbf{a},t)$ and initial density is put to equal 1. Notice, that variation of the action with by the Lagrangian (8.1) relative to $\mathbf{x}(\mathbf{a})$ gives the equation of motion (6.2) (or the equivalent equation for vector $\mathbf{u}$ (6.3)).

Due to the presence of magnetic field in the Lagrangian (8.1), the relabeling symmetry, in comparison with ideal hydrodynamics, reduces. If the first two terms in (8.1) are invariant with respect to all incompressible changes $\mathbf{a} \to \mathbf{a}(\mathbf{b})$ with $J|_b = 1$, invariance of the last term, however, restricts the class of possible deformations up to the following class

$$(\mathbf{h_0}(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{b} = \mathbf{h_0}(\mathbf{b}).$$

For infinitesimal transformations

$$\mathbf{a} \to \ \mathbf{a} + \tau\mathbf{g}(\mathbf{a})$$

where τ is a (small) group parameter the vector $\mathbf{g}$ must satisfy two conditions:

$$\mathrm{div}_a\mathbf{g} = 0, \ \ \mathrm{curl}_a[\mathbf{g} \times \mathbf{h_0}] = 0. \tag{8.2}$$

The first condition is the same as for ideal hydrodynamics, the second one provides conservation of magnetic field frozenness.

The conservation laws generating by this symmetry, in accordance with Noether theorem, can be obtained by standard scheme from the Lagrangian (8.1). They are written through the infinitesimal deformation $\mathbf{g}(\mathbf{a})$ as integral over $\mathbf{a}$:

$$I = \int (\mathbf{u}, \mathbf{g}(\mathbf{a}))d\mathbf{a} \tag{8.3}$$

where the vector $\mathbf{u}$ is given by (2.6). Putting $\mathbf{g} = \mathbf{h_0}$ from this (infinite) family of integrals one gets the simplest one

$$I_{ch} = \int (\mathbf{v}, \mathbf{h})d\mathbf{r}$$

which represents a cross-helicity characterizing degree of mutual knottiness of vortex and magnetic lines.

The conservation laws (8.3) are compatible with the Weber-type transformation. Really, substituting (6.4) into (8.3) and using (8.2) one leads to the relation

$$\int (\mathbf{u_0}(\mathbf{a}), \mathbf{g}(\mathbf{a}))d\mathbf{a}.$$

Hence conservation of (8.3) also follows. Note that if one would not suppose an independence of $\mathbf{u_0}$ on t then, due to arbitrariness of $\mathbf{g}(\mathbf{a})$, this could be considered as independent verification of conservation of solenoidal field $\mathbf{u_0}$:

$$\frac{d}{dt}\mathbf{u_0} = 0.$$

The MHD equations expressed in terms of Lagrangian variables become Hamiltonian ones, as in usual mechanics, for momentum $\mathbf{p} = \dot{\mathbf{x}}$ and coordinate $\mathbf{x}$. These variables assign the canonical Poisson structure.

In the Eulerian representation the MHD equations can be written also in the Hamiltonian form [14]:

$$\rho_t = \{\rho, H\}, \;\; \mathbf{v}_t = \{\mathbf{v}, H\}, \;\; \mathbf{h}_t = \{\mathbf{h}, H\},$$

where noncanonical Poisson bracket $\{F, G\}$ is given by the expression (1.9). As for ideal hydrodynamics, this Poisson bracket occurs to be degenerated. For example, the cross helicity I_{ch} serves a Casimir for the bracket (1.9). The reason of the Poisson bracket degeneracy is the same as for one-fluid hydrodynamics - it is connected with a relabeling symmetry of Lagrangian markers.

For incompressible case the Poisson bracket (1.9) reduces so that it can be expressed only through magnetic field $\mathbf{h}$ and vorticity $\mathbf{\Omega}$:

$$\{F, G\} = \int \left(\frac{\mathbf{h}}{\rho} \cdot \left(\left[\mathrm{curl}\frac{\delta F}{\delta \mathbf{h}} \times \mathrm{curl}\frac{\delta G}{\delta \mathbf{\Omega}}\right] - \left[\mathrm{curl}\frac{\delta G}{\delta \mathbf{h}} \times \mathrm{curl}\frac{\delta F}{\delta \mathbf{\Omega}}\right]\right)\right) d^3\mathbf{r} \tag{8.4}$$

$$+ \int \left(\mathbf{\Omega}\left[\mathrm{curl}\frac{\delta F}{\delta \mathbf{\Omega}} \times \mathrm{curl}\frac{\delta G}{\delta \mathbf{\Omega}}\right]\right) d^3\mathbf{r}.$$

This bracket remains also degenerated.

9 Variational principle for incompressible MHD

By analogy with incompressible hydrodynamics, one can introduce magnetic line representation:

$$\mathbf{h}(\mathbf{r}, t) = \int \delta(\mathbf{r} - \mathbf{R}(\mathbf{a}, t))(\mathbf{h}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a}, t) d^3\mathbf{a}. \tag{9.1}$$

For vorticity the analog of vortex line parametrization (3.4) can be obtained, for instance, as a limit $\epsilon \to 0$ of the corresponding representations for the two-fluid system. Calculations give [21]:

$$\mathbf{\Omega}(\mathbf{r}, t) = \int \delta(\mathbf{r} - \mathbf{R}(\mathbf{a}, t))((\mathbf{\Omega}_0(\mathbf{a}) + \mathrm{curl}_{\mathbf{a}}[\mathbf{h}_0(\mathbf{a}) \times \mathbf{U}(\mathbf{a}, t)])\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a}, t) d^3\mathbf{a}, \tag{9.2}$$

Here the field $\mathbf{U}(\mathbf{a}, t)$ is not assumed solenoidal, as well as the Jacobian of mapping $\mathbf{r} = \mathbf{R}(\mathbf{a}, t)$ is not equal to unity.

From the corresponding limit of the two-fluid system to incompressible MHD it is possible also to get the expression for Lagrangian

$$L = \int d^3\mathbf{a}([(\mathbf{h}_0\nabla_{\mathbf{a}})\mathbf{R} \times (\mathbf{U}\nabla_{\mathbf{a}})\mathbf{R}] \cdot \mathbf{R}_t) + \tag{9.3}$$

$$+1/3\int d^3\mathbf{a}([\mathbf{R}_t\times\mathbf{R}]\cdot(\mathbf{\Omega}_0\nabla_{\mathbf{a}})\mathbf{R})-\mathcal{H}\{\mathbf{\Omega}\{\mathbf{R},\mathbf{U}\},\mathbf{h}\{\mathbf{R}\}\}.$$

The Hamiltonian of the incompressible MHD $\mathcal{H}_{MHD}$ in terms of $\mathbf{U}(\mathbf{a},t)$ and $\mathbf{R}(\mathbf{a},t)$ takes the form

$$\mathcal{H}_{MHD}=\frac{1}{8\pi}\int\frac{((\mathbf{h}_0(\mathbf{a})\nabla_{\mathbf{a}})\mathbf{R}(\mathbf{a}))^2}{\det||\partial\mathbf{R}/\partial\mathbf{a}||}d^3\mathbf{a}+$$

$$+\frac{1}{8\pi}\int\int\frac{((\mathbf{\Omega}(\mathbf{a}_1)\nabla_1)\mathbf{R}(\mathbf{a}_1)\cdot(\mathbf{\Omega}(\mathbf{a}_2)\nabla_2)\mathbf{R}(\mathbf{a}_2))}{|\mathbf{R}(\mathbf{a}_1)-\mathbf{R}(\mathbf{a}_2)|}d^3\mathbf{a}_1d^3\mathbf{a}_2, \tag{9.4}$$

where we introduce the notation

$$\mathbf{\Omega}(\mathbf{a},t)=\mathbf{\Omega}_0(\mathbf{a})+\mathrm{curl}_{\mathbf{a}}[\mathbf{h}_0(\mathbf{a})\times\mathbf{U}(\mathbf{a},t)].$$

Equations of motion for $\mathbf{U}$ and $\mathbf{R}$ follow from the variational principle for action with Lagrangian (9.3):

$$[(\mathbf{h}_0\nabla_{\mathbf{a}})\mathbf{R}\times\mathbf{R}_t]\cdot(\partial\mathbf{R}/\partial a_\lambda)=-\delta\mathcal{H}/\delta U_\lambda, \tag{9.5}$$

$$[(\mathbf{\Omega}(\mathbf{a},t)\nabla_{\mathbf{a}})\mathbf{R}\times\mathbf{R}_t]-[(\mathbf{h}_0\nabla_{\mathbf{a}})\mathbf{R}\times(\mathbf{U}_t\nabla_{\mathbf{a}})\mathbf{R}]=\delta\mathcal{H}/\delta\mathbf{R}. \tag{9.6}$$

These equations can be obtained also directly from the MHD system (1.5-1.7) by the same scheme as it was done for ideal hydrodynamics.

Thus, we have variational principle for the MHD-type equations for two solenoidal vector fields. Their topological properties are fixed by $\mathbf{\Omega}_0(\mathbf{a})$ and $\mathbf{h}_0(\mathbf{a})$. These quantities represent Casimirs for the initial Poisson bracket (8.4). It is worth noting that the obtained equations of motion have the gauge invariant form. This gauge invariance is a remaining symmetry connected with relabeling of Lagrangian markers of magnetic lines in two-dimensional manifold which can be specified always locally. Coordinates of this manifold enumerate magnetic lines. This symmetry leads to conservation of volume of magnetic tubes including infinitesimally small magnetic tubes, namely, magnetic lines. This property explains why the Jacobian of the mapping $\mathbf{r}=\mathbf{R}(\mathbf{a},t)$ can be not equal identically to unity.

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References

[1] V.E.Zakharov and E.A.Kuznetsov, Doklady USSR Ac. Nauk. (Soviet Doklady), **194**, 1288 (1970).

[2] V.E.Zakharov and E.A.Kuznetsov, Uspekhi fizicheskikh nauk (Physics Uspekhi),**167**, 1137 (1997).

[3] H.Lamb, Hydrodynamics, Cambridge Univ. Press, 1932.

[4] E.A.Kuznetsov, A.V.Mikhailov, Phys. Lett., **77A**, 37 (1980).

[5] J.J. Moreau, C.R.Acad. Sc. Paris, 252, 2810 (1961); H.K.Moffatt, J. Fluid Mech. 35, 117 (1969).

[6] V.I.Arnold, Doklady Ac. Nauk SSSR, 162, 773-777 (1965) (in Russian).

[7] V.I.Arnold, Uspekhi matematicheskikh nauk, 24, N3, 225 (1969); Mathematical methods of classical mechanics, Moscow, Nauka, 1974 (in Russian).

[8] R.Salmon, Ann. Rev. Fluid Mech., **20**, 225 (1988).

[9] L.D.Landau and E.M.Lifshits, *Electrodynamics of continuous media*, Moscow, Nauka (1982).

[10] M.I. Monastyrskii and P.V.Sasorov, ZhETF (JETP), **93**, 1210, (1987).

[11] V.V.Yan'kov, ZhETF (JETP), **107**, 414,(1995).

[12] S.S.Moiseev, R.Z.Sagdeev, A,V.Tur, and V.V.Yanovskii, ZhETF (JETP), **83**, 215, (1982).

[13] B.N.Kuvshinov, FIzika Plazmy (Russian Plasma Phys.), **22**, 971, (1996).

[14] P.J.Morrison and J.M.Greene, Phys. Rev. Lett., **45**, 790 (1980)

[15] R.Hasimoto, J. Fluid Mech., **51**, 477 (1972).

[16] E.A.Kuznetsov and V.P.Ruban, Pis'ma v ZhETF, **67**, 1012 (1998) [JETP Letters, **67**, 1076 (1998)].

[17] M.Rasetti and T.Regge, Physica **80A**, 217 (1975).

[18] G.E.Volovik, V.S.Dotsenko (jun.), Pis'ma ZhETF (JETP Letters), **29**, 630 (1979).

[19] V.E.Zakharov and L.A.Takhtajan, Teor. Mat. Fiz., **38**, 26 (1979) (in Russian).

[20] V.I.Karpman, *Nonlinear waves in dispersive media*, Moscow, Nauka, (1973) (in Russian).

[21] V.P.Ruban, *Magnetic line motion in MHD*, Preprint of Landau Institute, JETP, submitted (1999).

[22] V.I.Il'gisonis and V.P.Pastukhov, Fizika Plazmy (Russian Plasma Phys.), **22**, 228 (1996).